

# QUANTUM TOROIDAL $\mathfrak{gl}_1$ AND BETHE ANSATZ

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*Dedicated to Rodney Baxter on the occasion of his 75th birthday*

**ABSTRACT.** We establish the method of Bethe ansatz for the XXZ type model obtained from the  $R$  matrix associated to quantum toroidal  $\mathfrak{gl}_1$ . We do that by using shuffle realizations of the modules and by showing that the Hamiltonian of the model is obtained from a simple multiplication operator by taking an appropriate quotient. We expect this approach to be applicable to a wide variety of models.

## 1. INTRODUCTION

The XXZ type models constitute a well-known large family of integrable quantum models, which was one of the main motivations for the very discovery of quantum groups.

These models arise in the following algebraic setting. We start with a quantum algebra with a triangular decomposition  $\mathcal{E} = \mathcal{E}_> \otimes \mathcal{E}_0 \otimes \mathcal{E}_<$ , and an associated  $R$  matrix  $\mathcal{R}$  in a completion of  $\mathcal{E}_\geq \otimes \mathcal{E}_\leq$ , where  $\mathcal{E}_\geq = \mathcal{E}_> \otimes \mathcal{E}_0$  and  $\mathcal{E}_\leq = \mathcal{E}_0 \otimes \mathcal{E}_<$ . We also fix a group like element  $t$  in (a completion of)  $\mathcal{E}$ . For an  $\mathcal{E}$  module  $U$ , we have the transfer matrix  $T_U(t) = (1 \otimes \text{Tr}_U)((1 \otimes t)\mathcal{R})$ , provided the trace is well-defined. The assignment  $U \mapsto T_U(t)$  gives us a map from the Grothendieck ring of a suitable category of  $\mathcal{E}$  modules to a completion of  $\mathcal{E}_\geq$ . The standard properties of the trace and the  $R$  matrix imply that this map is a ring homomorphism, and that the image is a commutative subalgebra. Given a suitable  $\mathcal{E}$  module  $V$  the commutative subalgebra of transfer matrices acts in  $V$  and produces the XXZ type Hamiltonians associated to  $\mathcal{E}$  and  $V$ .

The problem of diagonalizing the action of the XXZ type Hamiltonians has been extensively studied for more than 80 years. It is done almost exclusively by the Bethe ansatz method. The approach is always the same: one writes a candidate for the eigenvector depending on auxiliary parameters in some explicit form—the so called off-shell Bethe vector. Then one proves that if the parameters satisfy a system of algebraic equations, then the off-shell Bethe vector is indeed an eigenvector with an explicit eigenvalue. The system of equations is called Bethe equations and the corresponding eigenvector is called Bethe vector. Then, in good situations, one proves that the Bethe ansatz is complete, meaning that the Bethe vectors form a basis of the representation  $V$  modulo explicit symmetries if any.

In this paper, we study the case of  $\mathcal{E}$  being the quantum toroidal algebra of type  $\mathfrak{gl}_1$ , also known as elliptic Hall algebra,  $(q, \gamma)$  analog of  $\mathcal{W}_{1+\infty}$ , Ding-Iohara algebra, etc.. This algebra enjoys a wave of popularity due to its appearance in geometry [BS], [FT], [S], [SV1], [SV2] and in integrable systems [FKSW], [FKSW2], [KS].

It appears that the known methods of finding off-shell Bethe vectors are not directly applicable to  $\mathcal{E}$ . We propose an alternative way to obtain the spectrum of the Hamiltonians. The idea is to introduce an appropriate space of functions, and to identify the Hamiltonians with the

projection of simple operators of multiplication by symmetric functions. The Bethe equations appear naturally as the condition for describing the kernel of the projection. We expect that this method can be applied to many cases including the ones where the standard Bethe ansatz technique is already established.

Let us describe the logic of our approach in more detail. The quantum toroidal  $\mathfrak{gl}_1$  algebra  $\mathcal{E}$  depends on complex parameters  $q_1, q_2, q_3$  such that  $q_1 q_2 q_3 = 1$ . The algebras  $\mathcal{E}_>$ ,  $\mathcal{E}_<$  and  $\mathcal{E}_0$  are generated by currents  $e(z)$ ,  $f(z)$  and  $\psi^\pm(z)$  (plus central elements and their duals) respectively. The commutation relations are similar to that of the quantum affine  $\mathfrak{sl}_2$  algebra, but they are written in terms of the cubic polynomial  $g(z, w) = (z - q_1 w)(z - q_2 w)(z - q_3 w)$ , see Section 2. There is a projective action of the group  $SL(2, \mathbb{Z})$  on  $\mathcal{E}$  by automorphisms. Along with the initial currents  $e(z)$ ,  $f(z)$  and  $\psi^\pm(z)$ , we also use the currents  $e^\perp(z)$ ,  $f^\perp(z)$  and  $\psi^{\pm, \perp}(z)$  obtained by applying the automorphism from  $SL(2, \mathbb{Z})$  corresponding to the rotation by 90 degrees [BS], [M]. We call them “perpendicular” currents.

We define the coproduct and the  $R$  matrix in terms of perpendicular currents. We consider modules which are lowest weight modules with respect to initial currents, namely modules generated by a vector  $|\emptyset\rangle$  such that

$$(1.1) \quad f(z)|0\rangle = 0, \quad \psi^\pm(z)|0\rangle = \phi(z)|0\rangle,$$

where  $\phi(z)$  is a rational function. The most important example is a family of Fock modules  $\mathcal{F}(u)$  depending on a complex parameter  $u$ . These modules are irreducible under the Heisenberg subalgebra of  $\mathcal{E}$  generated by perpendicular currents  $\psi^{\pm, \perp}(z)$ . Other perpendicular currents  $e^\perp(z)$ ,  $f^\perp(z)$  act in  $\mathcal{F}(u)$  by vertex operators [FKSW], while operators  $\psi^\pm(z)$  can be identified with Macdonald operators [FHHSY], [FFJMM2].

Among the Hamiltonians of the model, the simplest is the degree one term  $H_p$  of the transfer matrix  $T_{\mathcal{F}(u)}(p^{d^\perp})$ , where  $p \in \mathbb{C}$  and  $d^\perp$  is the degree operator counting  $e(z)$  as 1 and  $f(z)$  as  $-1$ , see Lemma 5.1. It turns out that  $H_p$  coincides with the operator considered in [FKSW], [FKSW2], [KS] in relation to the deformed Virasoro algebra. Operator  $H_p$  acting in a generic tensor product of Fock modules for generic  $p$  has simple spectrum, and we do not consider other Hamiltonians in the present paper.

Following the ideas of [FO], [Ng], we realize algebra  $\mathcal{E}_>$  in an appropriate space of functions  $Sh_0$ , see Section 3. We also introduce another space of functions  $Sh_1(u)$  together with left and right actions of algebra  $Sh_0$ . Moreover, we extend the left action to the action of  $\mathcal{E}$ . We denote  $J_0$  the image of the right action:  $J_0 = Sh_1(u)Sh'_0$ , where the prime denotes the augmentation ideal. We use certain filtration, see Appendix A, to prove that the quotient  $Sh_1(u)/J_0$  is isomorphic to the Fock module  $\mathcal{F}(u)$  as  $\mathcal{E}$  module. We introduce a subspace  $N$  of functions in  $Sh_1(u)$  defined by certain regularity conditions and show that  $N \oplus J_0 = Sh_1(u)$ , see Section 3.4. Under a natural embedding of  $Sh_1(u)$  to a completion of  $\mathcal{E}_\geq$ , the space  $N$  is identified with the space of matrix elements of  $L$  operators of the form  $L_{\emptyset, v} = (1 \otimes \langle \emptyset |) \mathcal{R}(1 \otimes v)$ ,  $v \in \mathcal{F}(u)$ . Moreover under the projection  $Sh_1(u) \rightarrow Sh_1(u)/J_0 = \mathcal{F}(u)$ , the function corresponding to  $L_{\emptyset, v}$  is mapped to  $v$ .

The coefficients of the series  $\psi^+(z)$  act in the space of functions  $Sh_1(u)$  by multiplications by symmetric polynomials. It is easy to see that  $H_0 = \lim_{p \rightarrow 0} H_p$  coincides (up to an explicit constant) with the linear term  $h_1$  of  $\psi^+(z)$ , and in particular that, in the subspace  $Sh_{1, n}(u)$  of functions in  $n$  variables,  $H_0$  acts simply by multiplication by  $\sum_{i=1}^n x_i$  (up to multiplicative and

additive explicit constants), see Theorem 5.2, (5.4), (5.1). In the limit  $p \rightarrow 0$  the algebra of all Hamiltonians of the model coincides with the algebra generated by coefficients of  $\psi^+(z)$ .

Finally, we define the space of  $p$ -commutators:  $J_p = \{Sh_1(u)g - p^{\deg g}gSh_1(u) \mid g \in Sh'_0\}$ . The multiplication by symmetric polynomials clearly preserves this space, and for generic  $p$  we have the direct sum decomposition of vector spaces:  $N \oplus J_p = Sh_1(u)$ .

Our principal result is: *the projection of operator  $H_0$  acting in  $Sh_1(u)$  to the space of matrix elements of  $L$  operators  $N$  along space  $J_p$  coincides with Hamiltonian  $H_p$  acting on  $\mathcal{F}(u) = N$ .* In other words,  $\text{Pr}_{J_0} H_p v = \text{Pr}_{J_p} H_0 v$  for all  $v \in N$ , see Theorem 5.2.

This identification immediately leads to the Bethe equation and the computation of the spectrum.

Namely, we consider the dual space to  $Sh_1(u)$  and evaluation functionals defined as evaluation of functions in  $Sh_1(u)$  at fixed complex numbers  $\{a_i\}$ . Such a functional is obviously an eigenvector with respect to multiplication by a function  $f$ , with the eigenvalue given by evaluation of  $f$  at  $\{a_i\}$ . Also clearly, the evaluation functional has  $J_p$  in the kernel if and only if the evaluation numbers  $\{a_i\}$  satisfy the Bethe equation

$$\phi(a_i) \prod_{j(\neq i)} \frac{g(a_j, a_i)}{g(a_i, a_j)} = p^{-1}$$

for all  $i$ , see (5.5) and Theorem 5.4, where  $\phi(z)$  is the weight of the module in (1.1). Therefore, we obtain a description of the spectrum of the Hamiltonians in the dual module  $V = \mathcal{F}(u)^*$ .

We also study the off-shell Bethe vector. A result of [FHSSY] allows us to write the canonical element of  $\mathcal{F}(u)^* \otimes \mathcal{F}(u)$  in the form  $\sum_\lambda \langle \lambda | \otimes f_\lambda(x) \in \mathcal{F}(u)^* \otimes N$  with explicit functions  $f_\lambda(x)$ . The latter is the off-shell Bethe vector, from which the Bethe vector is obtained by evaluating the second component at  $\{a_i\}$ . We give the result in Proposition 5.5.

In this paper we consider only tensor products of Fock spaces of  $\mathcal{E}$ , but we expect such a scheme can be used for many modules over many quantum algebras. Also we skip the question of the completeness of the Bethe ansatz here, but we expect it can be proved for generic  $p$  by deforming the  $p = 0$  evaluation maps  $\rho_\lambda$  described in Appendix A in a standard way.

We note that operators  $\psi^\pm(z)$  acting in  $\mathcal{F}(u)$  can be identified with operators acting in equivariant  $K$ -theory of the Hilbert scheme of points on  $\mathbb{C}^2$ , where  $q_1, q_3$  are equivariant parameters. Then the algebra of the XXZ type Hamiltonians  $\{T_V(p^{d^\perp})\}$  provides the deformation of these operators which is expected to be related to “quantum equivariant  $K$ -theory”. Such an interpretation was one of motivations for our work.

In the conformal limit, algebra  $\mathcal{E}$  becomes the  $W$  algebra, and the corresponding integrals of motion in relation with Bethe equations were studied in [L], [AL]. Another Hamiltonian of similar kind was considered in [Sa1], [Sa2]. We also feel that there is some connection to the work [NS], where the authors find a connection between Bethe ansatz and supersymmetric gauge theory.

The paper is constructed as follows. In Section 2 we describe algebraic properties of quantum toroidal  $\mathfrak{gl}_1$  algebra and the Fock module. In Section 3 we establish functional realizations of  $\mathcal{E}_>$  and the Fock module. For that we use Gordon filtration established in Appendix A. We study matrix elements of  $L$  operators and their relation to the shuffle algebras in Section 4. In Section

5 we describe the XXZ type Hamiltonians, compute explicitly the first one and diagonalize it. In Section 6 we extend our results to the case of tensor product of Fock modules.

## 2. QUANTUM TOROIDAL $\mathfrak{gl}_1$

In this section, we introduce our notation concerning the quantum toroidal  $\mathfrak{gl}_1$  algebra.

**2.1. Algebra  $\mathcal{E}$ .** Fix complex numbers  $q, q_1, q_2, q_3$  satisfying  $q_2 = q^2$  and  $q_1 q_2 q_3 = 1$ . We assume further that, for integers  $l, m, n \in \mathbb{Z}$ ,  $q_1^l q_2^m q_3^n = 1$  holds only if  $l = m = n$ . We set

$$g(z, w) = (z - q_1 w)(z - q_2 w)(z - q_3 w),$$

$$\kappa_r = (1 - q_1^r)(1 - q_2^r)(1 - q_3^r).$$

The quantum toroidal algebra of type  $\mathfrak{gl}_1$ , which we denote by  $\mathcal{E}$ , is a  $\mathbb{C}$ -algebra generated by elements

$$e_k, f_k \quad (k \in \mathbb{Z}), \quad h_r \quad (r \in \mathbb{Z} \setminus \{0\})$$

and invertible elements

$$C, C^\perp, D, D^\perp,$$

subject to the relations given below. We write them in terms of the generating series

$$e(z) = \sum_{k \in \mathbb{Z}} e_k z^{-k}, \quad f(z) = \sum_{k \in \mathbb{Z}} f_k z^{-k},$$

$$\psi^\pm(z) = (C^\perp)^{\mp 1} \exp\left(\sum_{r=1}^{\infty} \kappa_r h_{\pm r} z^{\mp r}\right).$$

The defining relations of  $\mathcal{E}$  read as follows.

$$C, C^\perp \text{ are central, } DD^\perp = D^\perp D,$$

$$De(z) = e(qz)D, \quad Df(z) = f(qz)D, \quad D\psi^\pm(z) = \psi^\pm(qz)D,$$

$$D^\perp e(z) = qe(z)D^\perp, \quad D^\perp f(z) = q^{-1}f(z)D^\perp, \quad D^\perp \psi^\pm(z) = \psi^\pm(z)D^\perp,$$

$$\psi^\pm(z)\psi^\pm(w) = \psi^\pm(w)\psi^\pm(z),$$

$$\frac{g(C^{-1}z, w)}{g(Cz, w)}\psi^-(z)\psi^+(w) = \frac{g(w, C^{-1}z)}{g(w, Cz)}\psi^+(w)\psi^-(z),$$

$$g(z, w)\psi^\pm(C^{(-1 \mp 1)/2}z)e(w) + g(w, z)e(w)\psi^\pm(C^{(-1 \mp 1)/2}z) = 0,$$

$$g(w, z)\psi^\pm(C^{(-1 \pm 1)/2}z)f(w) + g(z, w)f(w)\psi^\pm(C^{(-1 \pm 1)/2}z) = 0,$$

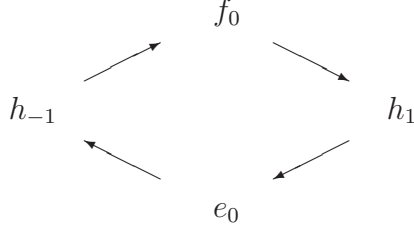
$$[e(z), f(w)] = \frac{1}{\kappa_1}\left(\delta\left(\frac{Cw}{z}\right)\psi^+(w) - \delta\left(\frac{Cz}{w}\right)\psi^-(z)\right),$$

$$g(z, w)e(z)e(w) + g(w, z)e(w)e(z) = 0,$$

$$g(w, z)f(z)f(w) + g(z, w)f(w)f(z) = 0,$$

$$\text{Sym}_{z_1, z_2, z_3} z_2 z_3^{-1} [e(z_1), [e(z_2), e(z_3)]] = 0,$$

$$\text{Sym}_{z_1, z_2, z_3} z_2 z_3^{-1} [f(z_1), [f(z_2), f(z_3)]] = 0.$$

FIGURE 1. Automorphism  $\theta$ .

In particular we have the relations

$$(2.1) \quad [h_r, e_n] = -\frac{1}{r} e_{n+r} C^{(-r-|r|)/2},$$

$$(2.2) \quad [h_r, f_n] = \frac{1}{r} f_{n+r} C^{(-r+|r|)/2},$$

$$(2.3) \quad [h_r, h_s] = \delta_{r+s,0} \frac{1}{r} \frac{C^r - C^{-r}}{\kappa_r},$$

for all  $r, s \in \mathbb{Z} \setminus \{0\}$  and  $n \in \mathbb{Z}$ .

The subalgebra of  $\mathcal{E}$  generated by  $e_n, f_n$  ( $n \in \mathbb{Z}$ ),  $h_r$  ( $r \in \mathbb{Z} \setminus \{0\}$ ) and  $C, C^\perp$  will be denoted by  $\mathcal{E}'$ .

Algebra  $\mathcal{E}$  admits an automorphism  $\theta$  of order 4 [BS, M] such that (see Fig.1)

$$(2.4) \quad \begin{aligned} \theta : e_0 &\mapsto h_{-1}, \quad h_{-1} \mapsto f_0, \quad f_0 \mapsto h_1, \quad h_1 \mapsto e_0, \\ C^\perp &\mapsto C, \quad C \mapsto (C^\perp)^{-1}, \quad D^\perp \mapsto D, \quad D \mapsto (D^\perp)^{-1}. \end{aligned}$$

Quite generally, we write  $x^\perp = \theta^{-1}(x)$  for an element  $x \in \mathcal{E}$ . In this notation

$$e_0^\perp = h_1, \quad f_0^\perp = h_{-1}, \quad h_1^\perp = f_0, \quad h_{-1}^\perp = e_0.$$

The relations (2.1)–(2.3) imply further that

$$\begin{aligned} e_1^\perp &= f_1 C^\perp, \quad e_{-1}^\perp = e_1 C^{-1}, \\ f_1^\perp &= f_{-1} C, \quad f_{-1}^\perp = e_{-1} (C^\perp)^{-1}, \\ e_{m+1}^\perp &= [e_m^\perp, f_0] C^\perp, \quad e_{-m-1}^\perp = [e_0, e_{-m}^\perp], \\ f_{m+1}^\perp &= [f_0, f_m^\perp], \quad f_{-m-1}^\perp = [f_{-m}^\perp, e_0] (C^\perp)^{-1}. \end{aligned}$$

Algebra  $\mathcal{E}$  is equipped with a  $\mathbb{Z}^2$  grading defined by the assignment

$$(2.5) \quad \deg e_n = (1, n), \quad \deg f_n = (-1, n), \quad \deg h_r = (0, r),$$

$$(2.6) \quad \deg x = (0, 0) \quad (x = C, C^\perp, D, D^\perp).$$

We have

$$\deg e_n^\perp = (-n, 1), \quad \deg f_n^\perp = (-n, -1), \quad \deg h_r^\perp = (-r, 0).$$

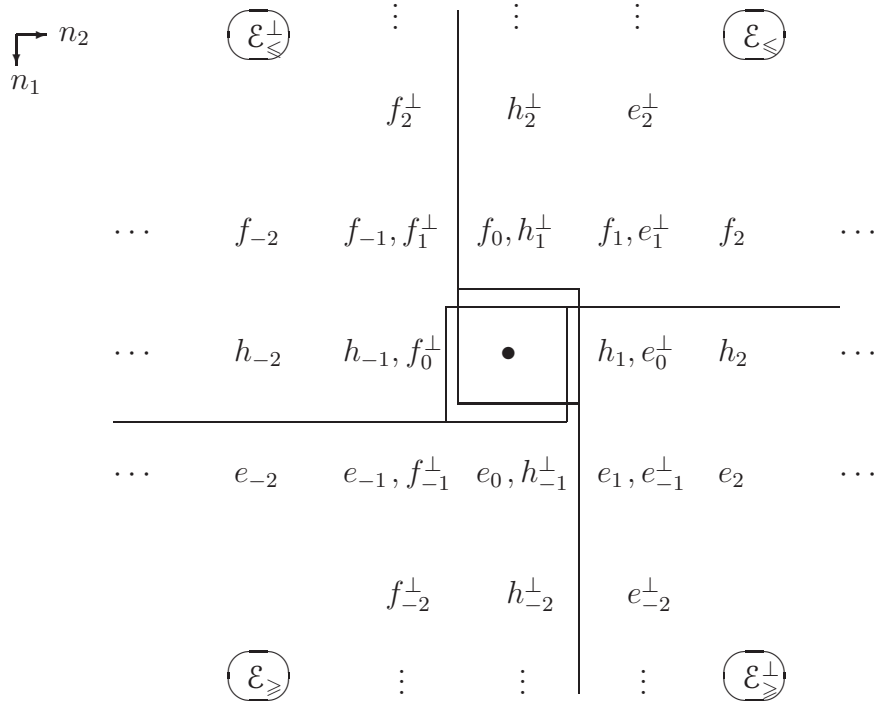


FIGURE 2. Subalgebras  $\mathcal{E}_{\geq}$ ,  $\mathcal{E}_{\leq}$ ,  $\mathcal{E}_{\geq}^{\perp}$ ,  $\mathcal{E}_{\leq}^{\perp}$ . The elements  $C, C^{\perp}, D, D^{\perp}$  placed at the center  $\bullet$  are common to all these subalgebras.

For a homogeneous element  $x \in \mathcal{E}$  with  $\deg x = (n_1, n_2)$ , we say that  $x$  has *principal degree*  $n_1$  and *homogeneous degree*  $n_2$  and write

$$\text{pdeg } x = n_1, \quad \text{hdeg } x = n_2.$$

Note that  $D^{\perp}x(D^{\perp})^{-1} = q^{\text{pdeg } x}x$  and  $DxD^{-1} = q^{-\text{hdeg } x}x$ .

Introduce the following subalgebras:

$$\begin{aligned} \mathcal{E}_{\geq} &= \langle e_n \ (n \in \mathbb{Z}), \ h_r \ (r > 0), \ C, C^{\perp}, D, D^{\perp} \rangle, \\ \mathcal{E}_{\leq} &= \langle f_n \ (n \in \mathbb{Z}), \ h_{-r} \ (r > 0), \ C, C^{\perp}, D, D^{\perp} \rangle, \\ \mathcal{E}_{\geq}^{\perp} &= \langle e_n^{\perp} \ (n \in \mathbb{Z}), \ h_r^{\perp} \ (r > 0), \ C, C^{\perp}, D, D^{\perp} \rangle, \\ \mathcal{E}_{\leq}^{\perp} &= \langle f_n^{\perp} \ (n \in \mathbb{Z}), \ h_{-r}^{\perp} \ (r > 0), \ C, C^{\perp}, D, D^{\perp} \rangle. \end{aligned}$$

We picture generators of algebra  $\mathcal{E}$  and their perpendicular counterparts on a plane according to their grading. The subalgebras  $\mathcal{E}_{\geq}, \mathcal{E}_{\leq}, \mathcal{E}_{\geq}^{\perp}, \mathcal{E}_{\leq}^{\perp}$  are generated by the elements appearing respectively in the lower, upper, right and left half plane, see Fig. 2.

We set also

$$\begin{aligned} \mathcal{E}_{>} &= \langle e_n \ (n \in \mathbb{Z}) \rangle, \quad \mathcal{E}_{<} = \langle f_n \ (n \in \mathbb{Z}) \rangle, \\ \mathcal{E}_{>}^{\perp} &= \langle e_n^{\perp} \ (n \in \mathbb{Z}) \rangle, \quad \mathcal{E}_{<}^{\perp} = \langle f_n^{\perp} \ (n \in \mathbb{Z}) \rangle. \end{aligned}$$

One can easily check that  $h_{-r}^{\perp}, Ce_{-r}^{\perp} \in \mathcal{E}_{>}$  for  $r > 0$ .

**2.2. Bialgebra structure and  $R$  matrix.** Algebra  $\mathcal{E}$  is endowed with a topological bialgebra structure. We choose the following coproduct  $\Delta$  and counit  $\varepsilon$ , defined in terms of the perpendicular generators,

$$(2.7) \quad \Delta(e_n^\perp) = \sum_{j \geq 0} e_{n-j}^\perp \otimes \psi_j^{+, \perp} (C^\perp)^n + 1 \otimes e_n^\perp,$$

$$(2.8) \quad \Delta(f_n^\perp) = f_n^\perp \otimes 1 + \sum_{j \geq 0} \psi_{-j}^{-, \perp} (C^\perp)^n \otimes f_{n+j}^\perp,$$

$$(2.9) \quad \Delta h_r^\perp = h_r^\perp \otimes 1 + (C^\perp)^{-r} \otimes h_r^\perp,$$

$$(2.10) \quad \Delta h_{-r}^\perp = h_{-r}^\perp \otimes (C^\perp)^r + 1 \otimes h_{-r}^\perp,$$

$$(2.11) \quad \Delta x = x \otimes x \quad (x = C, C^\perp, D, D^\perp),$$

$$(2.12) \quad \varepsilon(e_n^\perp) = \varepsilon(f_n^\perp) = 0, \quad \varepsilon(h_{\pm r}^\perp) = 0, \quad \varepsilon(x) = 1 \quad (x = C, C^\perp, D, D^\perp),$$

for all  $n \in \mathbb{Z}$  and  $r > 0$ . Here we set  $\psi_j^{\pm, \perp}(z) = \sum_{\pm j \geq 0} \psi_j^{\pm, \perp} z^{-j}$ ,  $\psi_0^{\pm, \perp} = C^{\pm 1}$ .

Quite generally, a bialgebra pairing on a bialgebra  $A$  is a symmetric non-degenerate bilinear form  $(\ , \ ) : A \times A \rightarrow \mathbb{C}$  with the properties

$$(a, b_1 b_2) = (\Delta(a), b_1 \otimes b_2), \quad (a, 1) = \varepsilon(a)$$

for any  $a, b_1, b_2 \in A$ . With each such pair  $(A, (\ , \ ))$ , there is an associated bialgebra  $DA$  called the Drinfeld double of  $A$ . As a vector space  $DA = A \otimes A^{\text{op}}$ , where  $A^{\text{op}}$  is a copy of  $A$  endowed with the opposite coalgebra structure. Moreover  $A^+ = A \otimes 1$  and  $A^- = 1 \otimes A^{\text{op}}$  are sub bialgebras of  $DA$ , and the commutation relation

$$\sum (a_{(2)}, b_{(1)}) a_{(1)}^- b_{(2)}^+ = \sum (b_{(2)}, a_{(1)}) b_{(1)}^+ a_{(2)}^-$$

is imposed for  $a, b \in A$ . Here  $a^+ = a \otimes 1$ ,  $a^- = 1 \otimes a$ , and we use the Sweedler notation  $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$  for the coproduct. The canonical element of  $DA = A \otimes A^{\text{op}}$  considered as an element of a suitable completion of  $A^+ \otimes A^- \subset DA \otimes DA$  is called the universal  $R$  matrix and is denoted by  $\mathcal{R}$ . It has the properties

$$(2.13) \quad \mathcal{R} \Delta(x) = \Delta^{\text{op}}(x) \mathcal{R} \quad (x \in DA),$$

$$(2.14) \quad (\Delta \otimes \text{id}) \mathcal{R} = \mathcal{R}_{1,3} \mathcal{R}_{2,3}, \quad (\text{id} \otimes \Delta) \mathcal{R} = \mathcal{R}_{1,3} \mathcal{R}_{1,2},$$

$$(2.15) \quad \mathcal{R}_{1,2} \mathcal{R}_{1,3} \mathcal{R}_{2,3} = \mathcal{R}_{2,3} \mathcal{R}_{1,3} \mathcal{R}_{1,2},$$

where, as usual, the suffixes  $i, j$  of  $\mathcal{R}_{i,j}$  stand for the tensor components, e.g.,  $\mathcal{R}_{1,2} = \mathcal{R} \otimes 1$ .

The bialgebra  $A = \mathcal{E}_{\geq}^\perp$  has a bialgebra pairing such that the non-trivial pairings of the generators are given by

$$(e_m^\perp, e_n^\perp) = \frac{1}{\kappa_1} \delta_{m,n}, \quad (h_r^\perp, h_s^\perp) = \frac{1}{r \kappa_r} \delta_{r,s},$$

$$(C, D) = (C^\perp, D^\perp) = q^{-1}.$$

This pairing respects the  $\mathbb{Z}^2$  grading in the sense that  $(a, b) = 0$  unless  $\deg a = \deg b$ . We identify  $A^{\text{op}}$  with  $\mathcal{E}_{\leq}^\perp$  through the following isomorphism of algebras, which is also an anti-isomorphism

of coalgebras,

$$e_n^\perp \mapsto f_{-n}^\perp, \quad h_r^\perp \mapsto h_{-r}^\perp, \quad x \mapsto x^{-1} \quad (x = C, C^\perp, D, D^\perp).$$

The Drinfeld double of  $\mathcal{E}_{\geq}^\perp$  is then identified with  $\mathcal{E}_{\geq}^\perp \otimes \mathcal{E}_{\leq}^\perp$ . Its quotient by the relation  $x \otimes 1 = 1 \otimes x$  ( $x = C, C^\perp, D, D^\perp$ ) is isomorphic to the algebra  $\mathcal{E}$  [BS].

The universal  $R$  matrix is an element of a certain completion of  $\mathcal{E}_{\geq}^\perp \otimes \mathcal{E}_{\leq}^\perp \subset \mathcal{E} \otimes \mathcal{E}$ , with the structure

$$(2.16) \quad \mathcal{R} = \mathcal{R}^{(0)} \mathcal{R}^{(1)} \mathcal{R}^{(2)}.$$

Here

$$(2.17) \quad \mathcal{R}^{(1)} = \exp \left( \sum_{r \geq 1} r \kappa_r h_r^\perp \otimes h_{-r}^\perp \right),$$

and

$$(2.18) \quad \mathcal{R}^{(2)} = 1 + \kappa_1 \sum_{i \in \mathbb{Z}} e_i^\perp \otimes f_{-i}^\perp + \dots$$

is the canonical element of  $\mathcal{E}_{\geq}^\perp \otimes \mathcal{E}_{\leq}^\perp$ . In (2.18),  $\dots$  stands for terms whose first component has homogeneous degree  $\geq 2$ . The element  $\mathcal{R}^{(0)}$  is formally defined as

$$\begin{aligned} \mathcal{R}^{(0)} &= q^{-c \otimes d - d \otimes c - c^\perp \otimes d^\perp - d^\perp \otimes c^\perp}, \\ C &= q^c, C^\perp = q^{c^\perp}, D = q^d, D^\perp = q^{d^\perp}. \end{aligned}$$

The expression  $(\mathcal{R}^{(0)})^{-1} \Delta^{\text{op}}(x) \mathcal{R}^{(0)}$  has a well defined meaning, and the intertwining property (2.13) should be understood as

$$\mathcal{R}^{(1)} \mathcal{R}^{(2)} \Delta(x) = ((\mathcal{R}^{(0)})^{-1} \Delta^{\text{op}}(x) \mathcal{R}^{(0)}) \mathcal{R}^{(1)} \mathcal{R}^{(2)} \quad (x \in \mathcal{E}).$$

The element  $\mathcal{R}^{(0)}$  is well defined on tensor products of representations which are principally graded and on which  $c$  acts as 0. This is the case for all representations considered in this paper.

**2.3. Fock representations.** Let  $V$  be an  $\mathcal{E}'$  module, and let  $L, K \in \mathbb{C}^\times$ . We say that  $V$  has *level*  $(L, K)$  if the central element  $C$  acts as the scalar  $L$  and  $C^\perp$  as  $K$ . In this paper we consider only modules of level  $(1, K)$ . Then the operators  $h_r$  are mutually commutative on  $V$ . We say that  $V$  is *quasi-finite* if it is graded by the principal degree,  $V = \bigoplus_{n \in \mathbb{Z}} V_n$ , and  $\dim V_n < \infty$  for all  $n$ . We say it is bounded if  $V_n = 0$  for  $n \ll 0$ . For an  $\mathcal{E}'$  module  $V$  and  $u \in \mathbb{C}^\times$ , we denote by  $V(u)$  the pullback of  $V$  by the automorphism

$$s_u : e(z) \mapsto e(z/u), \quad f(z) \mapsto f(z/u), \quad \psi^\pm(z) \mapsto \psi^\pm(z/u), \quad C \mapsto C, \quad C^\perp \mapsto C^\perp.$$

Let  $\phi(z)$  be a rational function such that  $\phi(z)$  is regular at  $z = 0, \infty$  and  $\phi(0)\phi(\infty) = 1$ . We say that  $V$  is a lowest weight module with lowest weight  $\phi(z)$  if it is generated by a vector  $v$  which satisfies

$$f(z)v = 0, \quad \psi^\pm(z)v = \phi^\pm(z)v.$$

Here  $\phi^\pm(z)$  means the expansion of  $\phi(z)$  at  $z^{\mp 1} = 0$ . For each such  $\phi(z)$ , there exists a unique irreducible lowest weight module  $L_{\phi(z)}$  with lowest weight  $\phi(z)$ . Assigning degree 0 to  $v$  we have the principal grading  $L_{\phi(z)} = \bigoplus_{n=0}^\infty (L_{\phi(z)})_n$ , and  $L_{\phi(z)}$  is quasi-finite [M].



The most basic lowest weight  $\mathcal{E}'$  module is the Fock module. For  $u \in \mathbb{C}^\times$ , the Fock module  $\mathcal{F}(u)$  is defined to be the irreducible lowest weight  $\mathcal{E}'$  module with level  $(1, q)$  and lowest weight

$$(2.19) \quad \phi(u, z) = \frac{q^{-1} - qu/z}{1 - u/z}.$$

As a vector space,  $\mathcal{F}(u)$  has a basis  $\{|\lambda\rangle\}_{\lambda \in \mathcal{P}}$  labeled by all partitions.

We use the following convention for partitions. A partition is a sequence of non-negative integers  $\lambda = (\lambda_1, \lambda_2, \dots)$  such that  $\lambda_i \geq \lambda_{i+1}$  for all  $i \geq 1$ , and  $\lambda_i = 0$  for  $i$  large enough. In particular, we write  $\emptyset = (0, 0, 0, \dots)$ . The set of all partitions is denoted by  $\mathcal{P}$ . The dual partition  $\lambda'$  is given by  $\lambda'_i = \#\{j \mid \lambda_j \geq i\}$ . We set  $|\lambda| = \sum_{j \geq 1} \lambda_j$  for  $\lambda \in \mathcal{P}$ . For  $j \geq 1$  and  $\lambda \in \mathcal{P}$  we write  $\lambda + \mathbf{1}_j = (\lambda_1, \lambda_2, \dots, \lambda_j + 1, \dots)$ .

We call a pair of natural numbers  $(x, y)$  *convex corner* of  $\lambda$  if  $\lambda'_{y+1} < \lambda'_y = x$ , and *concave corner* of  $\lambda$  if  $\lambda'_y = x - 1$  and in addition  $y = 1$  or  $\lambda'_{y-1} > x - 1$ . We denote by  $CC(\lambda)$  and  $CV(\lambda)$  the set of concave and convex corners of  $\lambda$ , respectively.

Then the action of the generators is given as follows [FT]:

$$\begin{aligned} \langle \lambda + \mathbf{1}_j | e(z) | \lambda \rangle &= \prod_{s=1}^{j-1} \psi(q_1^{\lambda_s - \lambda_j - 1} q_3^{s-j}) \prod_{s=1}^{j-1} \psi(q_1^{\lambda_j - \lambda_s} q_3^{j-s}) \cdot \delta(q_1^{\lambda_j} q_3^{j-1} u/z), \\ \langle \lambda | f(z) | \lambda + \mathbf{1}_j \rangle &= \frac{q - q^{-1}}{\kappa_1} \prod_{s=j+1}^{\ell(\lambda)} \psi(q_1^{\lambda_s - \lambda_j - 1} q_3^{s-j}) \prod_{s=j+1}^{\ell(\lambda)+1} \psi(q_1^{\lambda_j - \lambda_s} q_3^{j-s}) \cdot \delta(q_1^{\lambda_j} q_3^{j-1} u/z), \\ \langle \lambda | \psi^\pm(z) | \lambda \rangle &= \prod_{(i,j) \in CV(\lambda)} \psi(q_3^i q_1^j q_2 u/z) \prod_{(i,j) \in CC(\lambda)} \psi(q_3^i q_1^j q_2^2 u/z)^{-1}. \end{aligned}$$

In the above, we set  $\psi(z) = (q - q^{-1}z)/(1 - z)$  and assume that  $\lambda, \lambda + \mathbf{1}_j \in \mathcal{P}$ . In all other cases the matrix elements are defined to be zero. In terms of the generators  $h_r$ , we have for  $r \in \mathbb{Z} \setminus \{0\}$

$$(2.20) \quad h_r |\emptyset\rangle = \gamma_r |\emptyset\rangle, \quad \gamma_r = \frac{1 - q_2^r}{r \kappa_r} u^r.$$

The generators  $h_r^\perp$  act as a Heisenberg algebra on  $\mathcal{F}(u)$ ,

$$(2.21) \quad [h_r^\perp, h_s^\perp] = \frac{q^r - q^{-r}}{r \kappa_r} \delta_{r+s, 0} \quad (r, s \in \mathbb{Z} \setminus \{0\}),$$

and  $\mathcal{F}(u)$  is an irreducible module over this Heisenberg algebra. The generators  $e^\perp(z), f^\perp(z)$  act by vertex operators,

$$(2.22) \quad e^\perp(z) = \frac{1 - q_2}{\kappa_1} u \exp \left( \sum_{r=1}^{\infty} \frac{\kappa_r}{1 - q_2^r} h_{-r}^\perp z^r \right) \exp \left( \sum_{r=1}^{\infty} \frac{q^r \kappa_r}{1 - q_2^r} h_r^\perp z^{-r} \right),$$

$$(2.23) \quad f^\perp(z) = \frac{1 - q_2^{-1}}{\kappa_1} u^{-1} \exp \left( - \sum_{r=1}^{\infty} \frac{q^r \kappa_r}{1 - q_2^r} h_{-r}^\perp z^r \right) \exp \left( - \sum_{r=1}^{\infty} \frac{q^{2r} \kappa_r}{1 - q_2^r} h_r^\perp z^{-r} \right).$$

### 3. SHUFFLE ALGEBRAS

It is known that the algebra  $\mathcal{E}_> = \langle e_n \ (n \in \mathbb{Z}) \rangle$  has a presentation in terms of certain algebra of rational functions called the shuffle algebra. In this section we introduce an extension of the shuffle algebra which gives a functional realization of the Fock modules.

**3.1. Algebra  $Sh_0$ .** First, let us recall the definition of the shuffle algebra

$$Sh_0 = \oplus_{n=0}^{\infty} Sh_{0,n}.$$

We set  $Sh_{0,0} = \mathbb{C}$ ,  $Sh_{0,1} = \mathbb{C}[x^{\pm 1}]$ . For  $n \geq 2$ ,  $Sh_{0,n}$  is the space of all symmetric rational functions of the form

$$F(x_1, \dots, x_n) = \frac{f(x_1, \dots, x_n)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)^2}, \quad f(x_1, \dots, x_n) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{\mathfrak{S}_n},$$

satisfying the *wheel condition*

$$(3.1) \quad f(x_1, \dots, x_n) = 0 \quad \text{if } (x_1, x_2, x_3) = (x, q_1 x, q_1 q_2 x) \text{ or } (x, q_2 x, q_1 q_2 x).$$

Note that since  $f(x_1, \dots, x_n)$  is symmetric, from (3.1), we also have  $f(x_1, \dots, x_n) = 0$  if  $(x_1, x_2, x_3) = (x, q_i x, q_i q_j x)$  or  $(x_1, x_2, x_3) = (x, q_i q_j x, q_i x)$  for  $i, j \in \{1, 2, 3\}$ ,  $i \neq j$ .

We define the shuffle product  $*$  of elements  $F \in Sh_{0,m}$  and  $G \in Sh_{0,n}$  by the formula

$$(F * G)(x_1, \dots, x_{m+n}) = \text{Sym} \left[ F(x_1, \dots, x_m) G(x_{m+1}, \dots, x_{m+n}) \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \omega(x_{m+j}, x_i) \right],$$

where

$$\omega(x, y) = \frac{(x - q_1 y)(x - q_2 y)(x - q_3 y)}{(x - y)^3} = \frac{g(x, y)}{(x - y)^3}.$$

Here and after we set

$$\text{Sym} f(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

It is easy to check that the space  $Sh_0$  becomes an associative algebra under the product  $*$ . The following fact is known (see [SV2], [FT], [Ng]).

**Proposition 3.1.** *Algebra  $Sh_0$  is generated by the subspace  $Sh_{0,1}$ . There is an isomorphism of algebras  $\sigma : \mathcal{E}_> \simeq Sh_0$  such that*

$$\sigma(e_i) = c_1 x^i \in Sh_{0,1}, \quad i \in \mathbb{Z},$$

where  $c_1 = q_3 / ((1 - q_1)(1 - q_3))$ .

Under the isomorphism above, the graded component  $(\mathcal{E}_>)_{n,d}$  corresponds to the subspace  $(Sh_{0,n})_d$  of  $Sh_{0,n}$  consisting of functions of homogeneous degree  $d \in \mathbb{Z}$ .

**3.2. The bimodule  $Sh_1(u)$ .** We fix  $u \in \mathbb{C}^\times$ , and consider a linear space

$$Sh_1(u) = \bigoplus_{n=0}^{\infty} Sh_{1,n}(u).$$

We set  $Sh_{1,0}(u) = \mathbb{C}$ ,  $Sh_{1,1}(u) = (x - u)^{-1} \mathbb{C}[x^{\pm 1}]$ . For  $n \geq 2$ ,  $Sh_{1,n}(u)$  is the space of all rational functions of the form

$$F(x_1, \dots, x_n) = \frac{f(x_1, \dots, x_n)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{i=1}^n (x_i - u)},$$

$$f(x_1, \dots, x_n) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{\mathfrak{S}_n},$$

such that they satisfy both the wheel condition (3.1) and an additional wheel condition

$$(3.2) \quad f(u, q_2 u, x_3, \dots, x_n) = 0.$$

In what follows, we denote the element  $1 \in Sh_{1,0}(u) = \mathbb{C}$  by  $\mathbf{1}$ .

For  $F \in Sh_{0,m}$  and  $G \in Sh_{1,n}(u)$ , we set

$$(F * G)(x_1, \dots, x_{m+n}) = \text{Sym} \left[ F(x_1, \dots, x_m) G(x_{m+1}, \dots, x_{m+n}) \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \omega(x_{m+j}, x_i) \prod_{i=1}^m \phi(u, x_i) \right],$$

$$(G * F)(x_1, \dots, x_{m+n}) = \text{Sym} \left[ G(x_{m+1}, \dots, x_{m+n}) F(x_1, \dots, x_m) \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \omega(x_i, x_{m+j}) \right],$$

where  $\phi(u, z)$  is given in (2.19). With this definition, by Proposition 3.1,  $Sh_1(u)$  is an  $\mathcal{E}_>$  bimodule. Later we will prove that  $Sh_1(u)$  is a cyclic bimodule, and  $\mathbf{1}$  is a cyclic vector (see Corollary 3.5).

We upgrade the left  $\mathcal{E}_>$  action to make  $Sh_1(u)$  a left  $\mathcal{E}'$  module of level  $(1, q)$ .

**Proposition 3.2.** *The following formula defines a left action of  $\mathcal{E}'$  on  $Sh_1(u)$ .*

$$(3.3) \quad \begin{aligned} e_k \cdot F &= c_1 x^k * F, \\ h_r \cdot F &= \left( -\frac{1}{r} \sum_{i=1}^n x_i^r + \gamma_r \right) F, \\ f_k \cdot F &= c_2 n \left( \text{res}_{z=0} + \text{res}_{z=\infty} \right) \frac{F(x_1, \dots, x_{n-1}, z) z^k dz}{\prod_{i=1}^{n-1} \omega(z, x_i)} \frac{dz}{z}. \end{aligned}$$

Here  $F \in Sh_{1,n}(u)$ ,  $k \in \mathbb{Z}$ ,  $r \in \mathbb{Z} \setminus \{0\}$ ,  $\gamma_r$  is defined in (2.20),  $c_1 = q_3 / ((1 - q_1)(1 - q_3))$  and  $c_2 = q_3^{-1} / (1 - q_2)$ . This action commutes with the right action of  $\mathcal{E}_>$ .

*Proof.* The proof is done by a direct computation. As an example we sketch the verification of the relation  $[e_k, f_l]F = (1/\kappa_1) \delta(z/w) (\psi^+(z) - \psi^-(z))$ . Let  $F \in Sh_{1,n}(u)$ ,  $k, l \in \mathbb{Z}$ . From the above definition we deduce that

$$[e_k, f_l]F = -\frac{1}{\kappa_1} \times \left( \text{res}_{z=0} + \text{res}_{z=\infty} \right) \prod_{i=1}^n \frac{\omega(x_i, z)}{\omega(z, x_i)} \cdot \phi(u, z) \cdot z^{k+l} \frac{dz}{z} \times F.$$

Comparing this with the expansions at  $z^{\pm 1} \rightarrow \infty$

$$\frac{\omega(x, z)}{\omega(z, x)} = \exp \left( - \sum_{\pm r > 0} \frac{1}{r} \kappa_r x^r z^{-r} \right), \quad \phi(u, z) = q^{\mp 1} \exp \left( \sum_{\pm r > 0} \kappa_r \gamma_r z^{-r} \right),$$

we obtain the desired relation. The rest of the relations can be checked similarly. In particular, the cubic Serre relations follow from the identity

$$\text{Sym}_{x_1, x_2, x_3} \frac{x_2}{x_3} (\omega_{3,1}\omega_{3,2}\omega_{2,1} - \omega_{3,1}\omega_{2,3}\omega_{2,1} - \omega_{1,3}\omega_{1,2}\omega_{3,2} + \omega_{1,2}\omega_{1,3}\omega_{2,3}) = 0,$$

where  $\omega_{i,j} = \omega(x_i, x_j)$ . □

**3.3. Functional realization of Fock module.** We consider the following left  $\mathcal{E}'$  submodule of  $Sh_1(u)$ ,

$$(3.4) \quad J_0 = \text{Span}_{\mathbb{C}}\{G * F \mid G \in Sh_1(u), F \in Sh_{0,n}, n \geq 1\} \subset Sh_1(u).$$

The following gives a realization of the Fock module as a quotient of a space of rational functions.

**Proposition 3.3.** *We have the isomorphism of left  $\mathcal{E}'$  modules  $Sh_1(u)/J_0 \simeq \mathcal{F}(u)$ .*

*Proof.* The module  $Sh_1(u)/J_0$  contains the lowest weight vector  $\mathbf{1}$  with the same lowest weight (2.19) as the Fock module. Hence, in order to prove the isomorphism, it is sufficient to show that each of its graded component has the same dimension as that of the Fock module. We show this in Appendix A, Corollary A.2. □

Therefore we have the canonical projection map  $\pi$ .

**Corollary 3.4.** *There exists a unique surjective homomorphism of left  $\mathcal{E}'$  modules*

$$(3.5) \quad \pi : Sh_1(u) \rightarrow \mathcal{F}(u), \quad \mathbf{1} \rightarrow |\emptyset\rangle$$

*which factorizes through  $Sh_1(u)/J_0$ .*

**3.4. The subspace  $N$ .** We have a short exact sequence of left  $\mathcal{E}'$  modules

$$0 \rightarrow J_0 \rightarrow Sh_1(u) \rightarrow \mathcal{F}(u) \rightarrow 0.$$

In this section, we split this sequence in the category of vector spaces. The reason for the choice of this particular splitting will be clarified later (see (4.1)).

Define a linear map  $\kappa : \mathcal{F}(u) \rightarrow Sh_1(u)$  by the requirements  $\kappa(|\emptyset\rangle) = \mathbf{1}$  and

$$(3.6) \quad \kappa(h_{-r}^\perp(v)) = h_{-r}^\perp \kappa(v) - q^r \kappa(v) h_{-r}^\perp,$$

for all  $r > 0$  and  $v \in \mathcal{F}(u)$ . Here we use the bimodule action of  $h_{-r}^\perp \in \mathcal{E}_>$ .

Since  $\mathcal{F}(u)$  is cyclic with respect to the algebra generated by  $h_{-r}^\perp, r > 0$ , the map  $\kappa$  is uniquely defined. We clearly have  $\pi\kappa = id$  and in particular,  $\kappa$  is injective. Let

$$(3.7) \quad N = \kappa(\mathcal{F}(u)) \subset Sh_1(u).$$

We clearly have a direct sum of vector spaces

$$(3.8) \quad Sh_1(u) = J_0 \oplus N.$$

Now we have the following.

**Corollary 3.5.** *The space  $Sh_1(u)$  is a cyclic  $\mathcal{E}_>$  bimodule with cyclic vector  $\mathbf{1}$ . It is a free  $\mathcal{E}_>$  right module generated by  $N$ . It is also a free  $\mathcal{E}_>$  left module generated by  $N$ .*

*Proof.* The first statement follows from Proposition 3.3. The right and left actions of  $Sh_0$  are clearly free. The corollary follows. □

The subspace  $N$  has a curious description in terms of regularity conditions.

We call a function  $G(x_1, \dots, x_n) \in Sh_1(u)$  regular at zero if there exists a well-defined limit  $\lim_{t \rightarrow 0} G(ty_1, \dots, ty_k, x_{k+1}, \dots, x_n)$ ,  $k = 1, \dots, n$ . We call a function  $G(x_1, \dots, x_n) \in Sh_1(u)$  regular at infinity if there exists a well-defined limit  $\lim_{t \rightarrow \infty} G(ty_1, \dots, ty_k, x_{k+1}, \dots, x_n)$ ,  $k = 1, \dots, n$ .

**Proposition 3.6.** *A function  $G(x_1, \dots, x_n) \in Sh_1(u)$  belongs to  $N$  if and only if it is regular at zero, regular at infinity and  $\lim_{t \rightarrow 0} G(ty_1, \dots, ty_n) = 0$ .*

*Proof.* It is known [Ng], that a function  $F(x_1, \dots, x_m) \in Sh_{0,m}$  belongs to the commutative algebra generated by  $h_{-r}^\perp$ ,  $r > 0$ , if and only if it is regular at zero and at infinity. Then it is easy to check that action (3.6) preserves the regularity and vanishing conditions described in the proposition. It is easy to check that action (3.6) preserves the regularity conditions at zero and at infinity. Noting  $\lim_{t \rightarrow 0} \phi(u, tx) = q$  we see further that the vanishing condition is also preserved. Since **1** satisfies these conditions, we obtain the only if part. For the if part, we compute the dimension of the space of functions using the same filtration as in [Ng].  $\square$

We remark that if one defined a map  $\tilde{\kappa} : \mathcal{F}(u) \rightarrow Sh_1(u)$  by changing  $q$  to  $q^{-1}$  in (3.6) then the image of  $\tilde{\kappa}$  would consist of functions  $G(x_1, \dots, x_n) \in Sh_1(u)$  which are regular at zero, regular at infinity and satisfy  $\lim_{t \rightarrow \infty} G(ty_1, \dots, ty_n) = 0$ .

#### 4. THE SUBSPACE OF MATRIX ELEMENTS OF $L$ OPERATORS

In this section we construct an inclusion of bimodule  $Sh_1(u)$  to a completion of algebra  $\mathcal{E}_{\geq}$ . Under this inclusion the subspace  $N \subset Sh_1(u)$ , see (3.7), has a description in terms of matrix elements of  $L$  operators.

**4.1. The matrix elements of  $L$  operators.** Let  $\mathcal{R}$  be the universal  $R$  matrix (2.16), and set  $\mathcal{R}' = q^{c \otimes d + d \otimes c} \mathcal{R}$ . For bounded quasi-finite modules  $V, W$ ,  $\mathcal{R}'$  gives a well defined operator on a tensor product  $V(u_1) \otimes W(u_2)$  for generic  $u_1, u_2$ .

For  $v \in \mathcal{F}(u)$  and  $w \in \mathcal{F}(u)^*$ , let

$$L_{w,v} = (1 \otimes w) \mathcal{R}' (1 \otimes v)$$

denote the matrix element of  $\mathcal{R}'$  with respect to the second component. We call elements  $L_{w,v}$  matrix elements of  $L$  operators.

We are mostly concerned with the case  $w = \langle \emptyset |$ . In what follows we abbreviate  $\langle \emptyset |$ ,  $|\emptyset \rangle$  simply as  $\emptyset$  in the index of the matrix elements of  $L$  operators.

If a coproduct of an element of  $\mathcal{E}$  is known, one can compute its commutation relations with the matrix elements of  $L$  operators. In particular, we have the following commutation relations with perpendicular generators which involve only matrix elements of  $L$  operators with  $w = \langle \emptyset |$ .

**Lemma 4.1.** *For all  $r, n > 0$  and  $v \in \mathcal{F}(u)$ , we have*

$$(4.1) \quad [h_{-r}^\perp, L_{\emptyset,v}]_{q^r} = L_{\emptyset, h_{-r}^\perp v},$$

$$(4.2) \quad [e_{-n}^\perp, L_{\emptyset,v}]_{q^{-n}} = L_{\emptyset, e_{-n}^\perp v} + q^{-n} \sum_{j \geq 1} L_{\emptyset, \psi_j^{+, \perp} v} \cdot e_{-n-j}^\perp.$$

In addition we have

$$(4.3) \quad [e_0^\perp, L_{\emptyset, v}] = L_{\emptyset, e_0^\perp v} - \gamma_1 L_{\emptyset, v} + \sum_{j \geq 1} L_{\emptyset, \psi_j^{+, \perp} v} \cdot e_{-j}^\perp,$$

$$(4.4) \quad [f_0^\perp, L_{\emptyset, v}] = L_{\emptyset, f_0^\perp v} - \gamma_{-1} L_{\emptyset, v} + \sum_{j \geq 1} L_{\emptyset, f_j^\perp v} \cdot \psi_{-j}^{-, \perp}.$$

Here we set  $[A, B]_p = AB - pBA$ .

*Proof.* The element  $\mathcal{R}'$  has the intertwining property

$$\begin{aligned} \mathcal{R}' \left( h_r^\perp \otimes 1 + (C^\perp)^{-r} \otimes h_r^\perp \right) &= (q^{-r} h_r^\perp \otimes 1 + 1 \otimes h_r^\perp) \mathcal{R}', \\ \mathcal{R}' \left( q^r h_{-r}^\perp \otimes 1 + 1 \otimes h_{-r}^\perp \right) &= (h_{-r}^\perp \otimes 1 + (C^\perp)^r \otimes h_{-r}^\perp) \mathcal{R}', \\ \mathcal{R}' \left( q^n e_n^\perp \otimes 1 + 1 \otimes e_n^\perp + q^n \sum_{j \geq 1} e_{n-j}^\perp \otimes \psi_j^{+, \perp} \right) \\ &= \left( e_n^\perp \otimes 1 + (C^\perp)^n \otimes e_n^\perp + \sum_{j \geq 1} (C^\perp)^n \psi_j^{+, \perp} \otimes e_{n-j}^\perp \right) \mathcal{R}', \\ \mathcal{R}' \left( f_n^\perp \otimes 1 + (C^\perp)^n \otimes f_n^\perp + \sum_{j \geq 1} (C^\perp)^n \psi_{-j}^{-, \perp} \otimes f_{n+j}^\perp \right) \\ &= \left( q^n f_n^\perp \otimes 1 + 1 \otimes f_n^\perp + q^n \sum_{j \geq 1} f_{n+j}^\perp \otimes \psi_{-j}^{-, \perp} \right) \mathcal{R}', \end{aligned}$$

where  $r > 0$  and  $n \in \mathbb{Z}$ . Taking the matrix element between  $\langle \emptyset |$  and  $v$  in the second component, we obtain the lemma.  $\square$

The above intertwining relations allow us to compute  $L_{\emptyset, \emptyset}$  explicitly.

**Proposition 4.2.** *The element  $L_{\emptyset, \emptyset}$  satisfies the commutation relations*

$$(4.5) \quad (z - u)e(z)L_{\emptyset, \emptyset} = (q^{-1}z - qu)L_{\emptyset, \emptyset}e(z),$$

$$(4.6) \quad (q^{-1}z - qu)f(z)L_{\emptyset, \emptyset} = (z - u)L_{\emptyset, \emptyset}f(z),$$

$$(4.7) \quad [h_r, L_{\emptyset, \emptyset}] = 0 \quad (\forall r \neq 0).$$

Explicitly we have

$$(4.8) \quad L_{\emptyset, \emptyset} = q^{-d^\perp} \exp \left( \sum_{r=1}^{\infty} (1 - q_2^{-r}) h_r u^{-r} \right).$$

*Proof.* In Lemma 4.1, we consider the case  $v = |\emptyset\rangle$ . We need two more formulas derived similarly:

$$\begin{aligned} [h_1^\perp, L_{\emptyset, \emptyset}]_q &= -q L_{\langle \emptyset | h_1^\perp, \emptyset}, \\ [e_1^\perp, L_{\emptyset, \emptyset}]_q &= -qu C^\perp L_{\langle \emptyset | h_1^\perp, \emptyset} - (1 - q^2) u h_1^\perp L_{\emptyset, \emptyset}. \end{aligned}$$

Using

$$\begin{aligned} e_0^\perp &= h_1, & e_1^\perp &= C^\perp f_1, & e_{-1}^\perp &= e_1, & h_1^\perp &= f_0, \\ f_0^\perp &= h_{-1}, & f_1^\perp &= f_{-1}, & f_{-1}^\perp &= (C^\perp)^{-1} e_{-1}, & h_{-1}^\perp &= e_0, \end{aligned}$$

and the relations

$$\begin{aligned} e_{-1}^\perp |\emptyset\rangle &= u h_{-1}^\perp |\emptyset\rangle, & f_{-1}^\perp |\emptyset\rangle &= (qu)^{-1} h_{-1}^\perp |\emptyset\rangle, \\ \langle \emptyset | e_1^\perp &= qu \langle \emptyset | h_1^\perp, & \langle \emptyset | f_1^\perp &= u^{-1} \langle \emptyset | h_1^\perp, \end{aligned}$$

which follow from (2.22)–(2.23), we find

$$\begin{aligned} [h_1, L_{\emptyset, \emptyset}] &= [h_{-1}, L_{\emptyset, \emptyset}] = 0, \\ (e_1 - u e_0) L_{\emptyset, \emptyset} &= L_{\emptyset, \emptyset} (q^{-1} e_1 - q u e_0), \\ (q^{-1} f_1 - q u f_0) L_{\emptyset, \emptyset} &= L_{\emptyset, \emptyset} (f_1 - u f_0). \end{aligned}$$

Taking commutators between the last two lines and  $h_{\pm 1}$ , we obtain (4.5) and (4.6).

Furthermore, (4.5) and (4.6) imply that

$$(z - u)(q^{-1}z - qu)\delta(z/w)[\psi^{+, \perp}(z) - \psi^{-, \perp}(z), L_{\emptyset, \emptyset}] = 0.$$

Let  $[\psi^{+, \perp}(z) - \psi^{-, \perp}(z), L_{\emptyset, \emptyset}] = \sum_{j \in \mathbb{Z}} X_j z^{-j}$ . Then

$$q^{-1}X_{j+1} - (q + q^{-1})uX_j + qu^2X_{j-1} = 0.$$

We have  $X_0 = 0$ , and we already know that  $X_{\pm 1} = 0$ . From this follows (4.7).

The unique element in the completion of  $\mathcal{E}_{\geq}^\perp$  with respect to the homogeneous degree hdeg when it becomes large satisfying (4.5)–(4.7) is given by (4.8). The lemma follows.  $\square$

We denote by  $\hat{\mathcal{E}}_{\geq}$  the completion of algebra  $\mathcal{E}_{\geq}$  with respect to the homogeneous degree hdeg when it becomes large.

**Corollary 4.3.** *We have  $L_{\emptyset, v} \in \hat{\mathcal{E}}_{\geq}$  for all  $v \in \mathcal{F}(u)$ .*

*Proof.* We have  $L_{\emptyset, \emptyset} \in \hat{\mathcal{E}}_{\geq}$  from (4.8). Since  $h_{-r}^\perp \in \mathcal{E}_{>}$  for  $r > 0$ , the corollary follows from (4.1).  $\square$

We denote  $\mathcal{N}$  the space of matrix elements of  $L$  operators with the first component  $\langle \emptyset |$ :

$$\mathcal{N} = \text{Span}_{\mathbb{C}}\{L_{\emptyset, v} \mid v \in \mathcal{F}(u)\} \subset \hat{\mathcal{E}}_{\geq}.$$

**4.2. Inclusion of the shuffle algebra to  $\hat{\mathcal{E}}_{\geq}$ .** Consider the  $\mathcal{E}_{>}$  bimodule

$$\mathcal{S}(u) = \mathcal{E}_{>} \cdot L_{\emptyset, \emptyset} \cdot \mathcal{E}_{>} \subset \hat{\mathcal{E}}_{\geq}.$$

Since  $L_{\emptyset, \emptyset}$  satisfies relations (4.5), there is a surjective map of  $\mathcal{E}_{>}$  bimodules

$$(4.9) \quad \iota : Sh_1(u) \rightarrow \mathcal{S}(u), \quad \mathbf{1} \mapsto L_{\emptyset, \emptyset}.$$

**Lemma 4.4.** *The map  $\iota$  in (4.9) is an isomorphism.*

*Proof.* It suffices to show that  $\iota$  is injective. Suppose that  $G = \sum_j F_j * \mathbf{1} * H_j$  is in the kernel of (4.9) where  $F_j, H_j \in Sh_0$ . Let  $a_j, b_j \in \mathcal{E}_{>}$  be the elements corresponding to  $F_j, H_j$ . In the completion of  $\mathcal{E}_{\geq}$  we have the commutation relation

$$e_n L_{\emptyset, \emptyset} = L_{\emptyset, \emptyset} \tilde{e}_n, \quad \tilde{e}_n = q e_n + (q - q^{-1}) \sum_{j \geq 1} u^{-j} e_{n+j}.$$

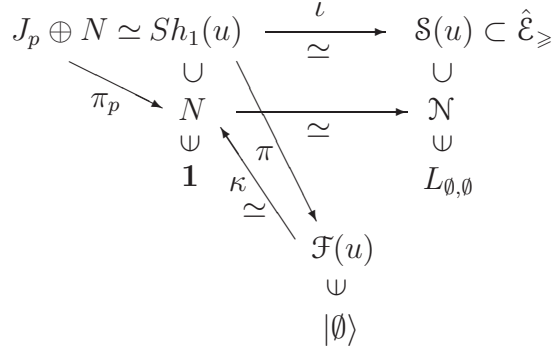


FIGURE 3. Subspaces isomorphic to Fock space

Using this we move the  $a_j$ 's to the right and obtain

$$0 = \sum_j a_j L_{\emptyset, \emptyset} b_j = \sum_j L_{\emptyset, \emptyset} \tilde{a}_j b_j,$$

where  $\tilde{a}_j$  is obtained from  $a_j$  by substituting  $e_n$  by  $\tilde{e}_n$ . Since  $L_{\emptyset, \emptyset}$  is invertible, this implies that  $\sum_j \tilde{a}_j b_j = 0$ . We may assume that  $G$  has principal degree, say,  $m$ . Let  $\tilde{G}$  be the element of the completion of  $Sh_0$  corresponding to  $\sum_j \tilde{a}_j b_j$ . Then we observe that  $\prod_{1 \leq i < j \leq m} (x_i - x_j)^2 \tilde{G}$  is nothing but the expansion of the rational function  $\prod_{1 \leq i < j \leq m} (x_i - x_j)^2 G$  at  $x_1 = \dots = x_m = 0$ . Hence we have  $G = 0$ .  $\square$

Then we have the identifications of spaces  $N$ ,  $\mathcal{N}$  and  $\mathcal{F}(u)$ .

**Lemma 4.5.** *We have  $\iota(N) = \mathcal{N} \subset \mathcal{S}(u)$ . For any  $v \in \mathcal{F}(u)$ , we have*

$$\pi \iota^{-1}(L_{\emptyset, v}) = v \in \mathcal{F}(u).$$

*Proof.* We have  $\pi \iota^{-1}(L_{\emptyset, \emptyset}) = |\emptyset\rangle \in \mathcal{F}(u)$ . The module  $\mathcal{F}(u)$  is cyclic with respect to the action of the  $h_{-r}^\perp$ 's. Formula (4.1) implies that  $h_{-r}^\perp L_{\emptyset, v} \equiv L_{\emptyset, h_{-r}^\perp v}$  holds modulo right action. Since the map  $\iota$  is  $\mathcal{E}_{>}$  linear and  $h_{-r}^\perp \in \mathcal{E}_{>}$ , we obtain the assertion.  $\square$

We capture various maps on Figure 3.

## 5. BETHE ANSATZ

**5.1. Integrals of motion.** Let  $W$  be a bounded quasi-finite module. We have  $W = \oplus_{n \in \mathbb{Z}} W_n$ , and  $d^\perp|_{W_n} = n$ . Fixing a parameter  $p \in \mathbb{C}^\times$ , consider the weighted trace

$$T_W(u; p) = \text{Tr}_{W(u), 2} \left( p^{1 \otimes d^\perp} \mathcal{R}' \right) = \sum_{n \in \mathbb{Z}} (pq^{-c^\perp})^n \text{Tr}_{W(u)_n, 2} \left( q^{-d^\perp \otimes c^\perp} \mathcal{R}^{(1)} \mathcal{R}^{(2)} \right),$$

where  $\mathcal{R}'$  is defined in Subsection 4.1, and  $\text{Tr}_{W(u), 2}$  signifies the trace in the second tensor component.

Note that the first tensor component of  $\mathcal{R}^{(1)}$  (2.17) has the homogeneous degree 0, and that of  $\mathcal{R}^{(2)}$  (2.18) has non-negative homogeneous degree. Therefore, the operator  $T_W(u; p)$  has



the form  $\sum_{l=0}^{\infty} T_{W,l}(p)u^{-l}$ , where  $T_{W,l}(p)$  is an operator on bounded quasi-finite modules and  $\deg T_{W,l}(p) = (0, l)$ .

We introduce the integrals of motion  $\{I_{W,l}(p)\}_{n=1}^{\infty}$  by

$$\log (T_{W,0}(p)^{-1}T_W(u;p)) = \sum_{n=1}^{\infty} I_{W,l}(p)u^{-l}.$$

For fixed  $p$ , the integrals of motions form a commutative family:

$$[T_{W_1,l_1}(p), T_{W_2,l_2}(p)] = 0,$$

for all  $l_1, l_2 \in \mathbb{Z}_{>0}$  and all bounded quasi-finite modules  $W_1, W_2$ .

When  $W(u)$  is the Fock module  $\mathcal{F}(u)$ , we have the following expression for  $I_{\mathcal{F},1}(p)$ .

**Lemma 5.1.** *Set  $\tilde{p} = pq^{-c^\perp}$ . Then the operator  $I_{\mathcal{F},1}(p)$  is given by*

$$(5.1) \quad I_{\mathcal{F},1}(p) = k \cdot \tilde{e}_0^\perp(p), \quad k = \frac{(q_1 q_3, \tilde{p}; \tilde{p})_\infty}{(\tilde{p} q_1, \tilde{p} q_3; \tilde{p})_\infty},$$

where  $\tilde{e}_0^\perp(p)$  is the coefficient of  $z^0$  of the twisted current

$$(5.2) \quad e^\perp(z; p) = e^\perp(z) \prod_{j=1}^{\infty} \psi^{+, \perp}(\tilde{p}^{-j} q^{-1} z),$$

and  $(a_1, \dots, a_m; \tilde{p})_\infty = \prod_{j=1}^m (a_j; \tilde{p})_\infty$ ,  $(a; \tilde{p})_\infty = \prod_{k=0}^{\infty} (1 - a\tilde{p}^k)$ .

*Proof.* From the definition of  $T_V(u, p)$  along with (2.17), (2.18), we obtain

$$\begin{aligned} T_{\mathcal{F},0}(p) &= q^{-d^\perp} \text{Tr}_{\mathcal{F}(u),2} \left[ \tilde{p}^{1 \otimes d^\perp} \mathcal{R}^{(1)} \right], \\ T_{\mathcal{F},1}(p) &= q^{-d^\perp} \kappa_1 \text{res}_{z=0} \text{Tr}_{\mathcal{F}(u),2} \left[ \tilde{p}^{1 \otimes d^\perp} \mathcal{R}^{(1)} \cdot 1 \otimes f^\perp(z) \right] e^\perp(z) \frac{dz}{z}. \end{aligned}$$

We then substitute (2.23) for  $f^\perp(z)$ . The trace can be calculated by using

$$\begin{aligned} &\text{Tr}_{\mathcal{F}(u),2} \left[ \tilde{p}^{1 \otimes d^\perp} \exp \left( \sum_{r=1}^{\infty} A_r 1 \otimes h_{-r}^\perp \right) \exp \left( \sum_{r=1}^{\infty} B_r 1 \otimes h_r^\perp \right) \right] \\ &= \frac{1}{(\tilde{p}; \tilde{p})_\infty} \exp \left( \sum_{r=1}^{\infty} A_r B_r \frac{\tilde{p}^r}{1 - \tilde{p}^r} \frac{q^r - q^{-r}}{r \kappa_r} \right), \end{aligned}$$

where we set

$$A_r = r \kappa_r h_r^\perp \otimes 1 - \frac{q^r \kappa_r}{1 - q^{2r}} z^r, \quad B_r = -\frac{q^{2r} \kappa_r}{1 - q^{2r}} z^{-r}.$$

Note that

$$\exp \left( \sum_{r=1}^{\infty} \kappa_r \frac{\tilde{p}^r q^r}{1 - \tilde{p}^r} h_r^\perp z^{-r} \right) = \prod_{j \geq 1} \psi^{+, \perp}(\tilde{p}^{-j} q^{-1} z).$$

After simplification we find

$$T_{\mathcal{F},0} = q^{-d^\perp} \frac{1}{(\tilde{p}; \tilde{p})_\infty}, \quad T_{\mathcal{F},1} = u^{-1} q^{-d^\perp} \frac{(q_2^{-1}; \tilde{p})_\infty}{(\tilde{p}q_1, \tilde{p}q_3; \tilde{p})_\infty} \tilde{e}_0^\perp(p).$$

The lemma follows.  $\square$

More generally, if  $W$  is a tensor product of several Fock modules, the operators  $\{I_{W,n}(p)\}_{n=1}^\infty$  are closely related to the commutative family of operators introduced and studied in [FKSW], [KS].

**5.2. Action of  $\tilde{e}_0^\perp(p)$  on Fock module as a projection.** In this subsection we identify the action of the integral of motion  $\tilde{e}_0^\perp(p)$  on the Fock module with a projection of operator  $h_1$  acting in  $Sh_1(u)$  to  $N$ , along the space  $J_p$ .

We fix  $p \in \mathbb{C}^\times$  and consider the subspace of  $Sh_1(u)$

$$J_p = \text{Span}_{\mathbb{C}}\{G * F - p^n F * G \mid G \in Sh_1(u), F \in Sh_{0,n}, n \geq 1\} \subset Sh_1(u).$$

Unlike (3.4), it is not an  $Sh_0$  submodule. However, it is clearly preserved by the action of  $h_r$ , see (3.3). From (3.8), we see that for generic  $p$ , we have a direct sum of vector spaces

$$(5.3) \quad Sh_1(u) = J_p \oplus N.$$

Denote  $\pi_p : Sh_1(u) \rightarrow N$  the projection operator in (5.3) along the first summand.

Recall that for  $G(x_1, \dots, x_n) \in Sh_{1,n}(u)$ , the action of  $h_1$  is simply given by

$$(5.4) \quad h_1 G(x_1, \dots, x_n) = \left(-\sum_{i=1}^n x_i + \gamma_1\right) G(x_1, \dots, x_n),$$

see (3.3). The crucial observation is that the projection of this simple operator to  $N$  along  $J_p$  produces the desired integral of motion  $\tilde{e}_0^\perp(p)$ .

**Theorem 5.2.** *Under the identification of  $N$  and  $\mathcal{F}(u)$ , we have  $\pi_p h_1 = \tilde{e}_0^\perp(p)$ . In other words, for any  $v \in \mathcal{F}(u)$  we have*

$$\tilde{e}_0^\perp(p)v = (\kappa^{-1} \circ \pi_p)(h_1 \kappa(v)).$$

*Proof.* We use the isomorphism  $\iota$ , see (4.9) and Lemma 4.5. We work in  $\mathcal{S}(u) \subset \hat{\mathcal{E}}_{\geq}$  and make use of the matrix elements of  $L$  operators to compute the projection.

By (4.2) and (4.3) we have

$$\begin{aligned} [e_0^\perp, L_{\emptyset,v}] + \gamma_1 L_{\emptyset,v} &= L_{\emptyset, e_0^\perp v} + \sum_{j \geq 1} L_{\emptyset, \psi_j^{+, \perp} v} \cdot e_{-j}^\perp \\ &\equiv L_{\emptyset, e_0^\perp v} + \sum_{j \geq 1} p^j e_{-j}^\perp \cdot L_{\emptyset, \psi_j^{+, \perp} v} \quad \text{mod } J_p, \end{aligned}$$

and for  $n > 0$

$$\begin{aligned} e_{-n}^\perp L_{\emptyset,v} &= L_{\emptyset, e_{-n}^\perp v} + q^{-n} \sum_{j \geq 0} L_{\emptyset, \psi_j^{+, \perp} v} \cdot e_{-j-n}^\perp \\ &\equiv L_{\emptyset, e_{-n}^\perp v} + \sum_{j \geq 0} q^{-n} p^{j+n} e_{-j-n}^\perp \cdot L_{\emptyset, \psi_j^{+, \perp} v} \quad \text{mod } J_p. \end{aligned}$$

Iterating the latter, we obtain

$$\begin{aligned} h_1 L_{\emptyset, v} &= [e_0^\perp, L_{\emptyset, v}] + \gamma_1 L_{\emptyset, v} \\ &\equiv L_{\emptyset, e_0^\perp v} + \sum_{\substack{k \geq 1 \\ j_k, \dots, j_2 \geq 0, j_1 \geq 1}} q^{j_k + \dots + j_1} (pq^{-1})^{j_k} \dots (pq^{-1})^{j_1} L_{\emptyset, e_{-j_k - \dots - j_1}^\perp \psi_{j_k}^{+, \perp} \dots \psi_{j_1}^{+, \perp} v} \pmod{J_p} \\ &= L_{\emptyset, \tilde{e}_0^\perp(p) v}, \end{aligned}$$

where we used definition (5.2).  $\square$

**5.3. Bethe ansatz.** Theorem 5.2 immediately leads to Bethe ansatz statements for the dual module.

Given a quasi-finite left  $\mathcal{E}'$  module  $V = \oplus_{n=0}^\infty V_n$ , we consider the graded dual space  $V^* = \oplus_{n=0}^\infty \text{Hom}(V_n, \mathbb{C})$ . As usual,  $V^*$  is a right  $\mathcal{E}'$  module with action given by  $gf(v) = f(gv)$ ,  $g \in \mathcal{E}'$ ,  $v \in V$ ,  $f \in V^*$ . Note that the spectrum of any operator and of the dual operator coincide. Note also that the dual to a lowest weight Fock left module is a highest weight Fock right module.

For a point  $a = (a_1, \dots, a_n) \in \mathbb{C}^n$  such that  $a_i \neq a_j$  ( $i \neq j$ ) and  $a_i \neq u$ , we denote by  $ev_a$  the evaluation map  $ev_a : Sh_1(u) \rightarrow \mathbb{C}$  defined by

$$ev_a(F(x_1, \dots, x_n)) = F(a_1, \dots, a_n), \quad F(x_1, \dots, x_n) \in Sh_{1,n}(u),$$

and  $ev_a(Sh_{1,m}(u)) = 0$  for  $m \neq n$ .

**Lemma 5.3.** *We have  $ev_a(J_p) = 0$  if and only if  $a = (a_1, \dots, a_n)$  satisfies the Bethe equation*

$$(5.5) \quad 1 = q^{-1} p \cdot \frac{a_i - q_2 u}{a_i - u} \prod_{j(\neq i)} \frac{(a_j - q_1 a_i)(a_j - q_2 a_i)(a_j - q_3 a_i)}{(a_j - q_1^{-1} a_i)(a_j - q_2^{-1} a_i)(a_j - q_3^{-1} a_i)}, \quad i = 1, \dots, n.$$

*Proof.* Let  $F \in Sh_{0,m}$ ,  $G \in Sh_{1,k}(u)$  and  $m \geq 1$ ,  $m + k = n$ . Then  $p^m F * G$  and  $G * F$  by definition are symmetrization and can be compared term-wise so that the substitutions of variables match. These terms become equal if and only if (5.5) holds.  $\square$

**Theorem 5.4.** *Let  $a = (a_1, \dots, a_n)$  be a solution of (5.5) such that  $ev_a$  is non-zero. Then the restriction of  $ev_a$  to  $N = \kappa(\mathcal{F}(u))$  is an eigenvector in  $\mathcal{F}(u)^*$  of the first integral of motion  $\tilde{e}_0^\perp(p)$  (5.2) with the eigenvalue*

$$E_1(a) = - \sum_{i=1}^n a_i + \gamma_1,$$

where we recall that  $\gamma_1 = u/((1 - q_1)(1 - q_3))$ . For generic  $p$ ,  $ev_a$  is a joint eigenvector of  $\{I_{\mathcal{F},n}(p)\}_{n=1}^\infty$ .

*Proof.* When  $p = 0$ , the operator  $I_{\mathcal{F},1}(p)$  on  $\mathcal{F}(u)$  has simple spectra. Hence it is enough to show that  $ev_a$  is an eigenvector of  $\tilde{e}_0^\perp(p)$  with eigenvalue  $E_1(a)$ .

We simply have for any  $G \in N$

$$\tilde{e}_0^\perp(p) ev_a(G) = ev_a(\tilde{e}_0^\perp(p) G) = ev_a(h_1 G) = E_1(a) ev_a(G),$$

where the second equality follows from Theorem 5.2 together with Lemma 5.3 and the third equality follows from (5.4).  $\square$

**5.4. The off-shell Bethe vector.** For us an off-shell Bethe vector is a vector depending on parameters  $a_i$  such that if  $a_i$  satisfy the Bethe equation, it becomes an eigenvector of Hamiltonians. However, such a requirement does not determine it uniquely.

An off-shell Bethe vector is obviously given by the formula  $(\text{id} \otimes ev_a \circ \kappa)K$ , where

$$K = \sum_{\lambda} \frac{\langle \emptyset | h_{\lambda}^{\perp} \otimes h_{-\lambda}^{\perp} | \emptyset \rangle}{\langle \emptyset | h_{\lambda}^{\perp} h_{-\lambda}^{\perp} | \emptyset \rangle}$$

is the canonical element of the space  $\mathcal{F}^*(u) \otimes \mathcal{F}(u)$ . Here and after, for a partition  $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)})$  we use the notation  $h_{\lambda}^{\perp} = h_{\lambda_1}^{\perp} \cdots h_{\lambda_{\ell(\lambda)}}^{\perp}$ , etc..

More generally, the off-shell Bethe vector is given by  $(\text{id} \otimes ev_a)(K_p + (1 \otimes \kappa)K)$  where  $K_p \in \mathcal{F}(u)^* \otimes J_p$ . Our goal is to give an explicit formula in terms of partitions for a suitable choice of  $K_p$ .

Let  $\Lambda$  denote the space of symmetric functions. For  $\lambda \in \mathcal{P}$ , let  $p_{\lambda}, m_{\lambda} \in \Lambda$  be the power sum and the monomial symmetric functions, respectively. Using the rescaled generators

$$\tilde{h}_r^{\perp} = r(1 - q_1^r)q_2^{r/2}q_3^r h_r^{\perp},$$

we identify the algebra generated by  $h_r^{\perp}$ ,  $r > 0$  with  $\Lambda$  by

$$\nu^* : \Lambda \xrightarrow{\sim} \mathbb{C}[h_r^{\perp}]_{r>0}, \quad p_{\lambda} \mapsto \tilde{h}_{\lambda}^{\perp}.$$

Introduce further the elements of  $Sh_0$

$$(5.6) \quad \epsilon_{\lambda}^{(q_3)} = \epsilon_{\lambda_1}^{(q_3)} * \cdots * \epsilon_{\lambda_{\ell(\lambda)}}^{(q_3)}, \quad \epsilon_n^{(q_3)}(x) = \prod_{i < j} \frac{(x_i - q_3 x_j)(x_i - q_3^{-1} x_j)}{(x_i - x_j)^2}.$$

Since for  $v \in \mathcal{F}(u)$  we have

$$L_{\emptyset, h_{-r}^{\perp}, v} = [h_{-r}^{\perp}, L_{\emptyset, v}]_{q^r} \equiv (1 - p^r q_2^{r/2}) h_{-r}^{\perp} L_{\emptyset, v} \text{ mod } J_p,$$

we obtain

$$(5.7) \quad (\text{id} \otimes \kappa)K \simeq \sum_{\lambda} \frac{\langle \emptyset | h_{\lambda}^{\perp} \rangle}{\langle \emptyset | h_{\lambda}^{\perp} h_{-\lambda}^{\perp} | \emptyset \rangle} \otimes \left( \prod_{i=1}^{\ell(\lambda)} (1 - p^{\lambda_i} q_2^{\lambda_i/2}) h_{-\lambda_i}^{\perp} * \mathbf{1} \right) = \sum_{\lambda} \frac{\langle \emptyset | \alpha(h_{\lambda}^{\perp}) \otimes \sigma(h_{-\lambda}^{\perp}) * \mathbf{1} \rangle}{\langle \emptyset | h_{\lambda}^{\perp} h_{-\lambda}^{\perp} | \emptyset \rangle},$$

where  $\alpha$  is an algebra homomorphism given by

$$(5.8) \quad \alpha(h_r^{\perp}) = (1 - p^r q_2^{r/2}) h_r^{\perp}.$$

Here  $\simeq$  denotes equality modulo vectors in  $\mathcal{F}(u)^* \otimes J_p$ .

Recall the isomorphism  $\sigma : \mathcal{E}_{>} \simeq Sh_0$  in Proposition 3.1. The following formula is known ([FHSSY], Proposition 1.12).

$$(5.9) \quad \sum_{\lambda \in \mathcal{P}} \frac{h_{\lambda}^{\perp} \otimes \sigma(h_{-\lambda}^{\perp})}{\langle \emptyset | h_{\lambda}^{\perp} h_{-\lambda}^{\perp} | \emptyset \rangle} = \sum_{\lambda \in \mathcal{P}} \frac{1}{(q_1 - 1)^{|\lambda|}} \frac{1}{\prod_{i=1}^{\ell(\lambda)} \lambda_i!} \nu^*(m_{\lambda}) \otimes \epsilon_{\lambda}^{(q_3)}.$$

Combining (5.9) and (5.7), we arrive at the following.

**Theorem 5.5.** *An off-shell Bethe vector in  $\mathcal{F}(u)^*$  is given by*

$$ev_a = \sum_{\lambda \in \mathcal{P}} \frac{1}{(q_1 - 1)^{|\lambda|}} \frac{1}{\prod_{i=1}^{\ell(\lambda)} \lambda_i!} ev_a(\epsilon_\lambda^{(q_3)}) \times \langle \emptyset | \alpha(\nu^*(m_\lambda)) \rangle.$$

Here  $\epsilon^{(q_3)}$  is given by (5.6),  $\nu^*(m_\lambda)$  denotes the element of  $\mathcal{F}^*(u)$  corresponding to the monomial symmetric function  $m_\lambda$ , and  $\alpha$  stands for the substitution (5.8).

## 6. BETHE ANSATZ IN TENSOR PRODUCT OF FOCK MODULES

The method described above for diagonalizing  $\tilde{e}_0^\perp(p)$  in the Fock module works for the case of other highest weight modules with straightforward modifications. Here we give some detail for the case of generic tensor products of Fock modules.

Consider  $V = \mathcal{F}(u_1) \otimes \mathcal{F}(u_2) \otimes \cdots \otimes \mathcal{F}(u_k)$ , where  $u_1, \dots, u_k \in \mathbb{C}^\times$  are generic numbers. For generic  $u_1, \dots, u_k$  this module is well defined and it is a bounded tame irreducible module, cf. [FFJMM2]. Set  $\mathbf{u} = (u_1, \dots, u_k)$  and denote the lowest weight vector of  $V$  by  $|\emptyset\rangle$ .

We define the corresponding shuffle algebra  $Sh_1(\mathbf{u}) = \bigoplus_{n=0}^\infty Sh_{1,n}(\mathbf{u})$ . The space  $Sh_{1,n}(\mathbf{u})$  consists of all rational functions of the form

$$F(x_1, \dots, x_n) = \frac{f(x_1, \dots, x_n)}{\prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{i=1}^n \prod_{j=1}^k (x_i - u_j)},$$

$$f(x_1, \dots, x_n) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{\mathfrak{S}_n},$$

such that they satisfy both the wheel condition (3.1) and additional wheel conditions

$$f(u_i, q_2 u_i, x_3, \dots, x_n) = 0, \quad i = 1, \dots, k.$$

We denote the element  $1 \in Sh_{1,0}(\mathbf{u}) = \mathbb{C}$  by  $\mathbf{1}$ .

Next we set

$$\phi(\mathbf{u}, x) = \prod_{i=1}^k \phi(u_i, x) = \prod_{i=1}^k \frac{qu_i - q^{-1}x}{u_i - x}$$

and define the left and the right action of  $Sh_0$  on  $Sh_1(\mathbf{u})$  by the same formulas as in the case  $k = 1$ .

Similarly, we extend the left action of  $Sh_0$  to the left action of  $\mathcal{E}'$ , define  $J_0 = Sh_1(\mathbf{u}) * Sh'_0$  and prove

**Proposition 6.1.** *We have an isomorphism  $Sh_1(\mathbf{u})/J_0 \simeq V$  of left  $\mathcal{E}'$  module sending  $\mathbf{1}$  to  $|\emptyset\rangle$ .*

*Proof.* The crucial part is to do the Gordon filtration to find the size of the quotient. It is done similarly to the case of  $k = 1$  discussed in Appendix A. The evaluation maps in the general case will depend on  $k$  partitions, each starts evaluation in  $u_i$ . Since  $u_i$  are generic, there is no interplay between different partitions.  $\square$

We define the map  $\pi$  to be the projection map.

For the definition of  $\kappa$  and the space  $N$ , it is not enough to use  $h_{-r}^\perp$  only since  $V$  is not a cyclic Heisenberg module. So, in addition to (3.6) we impose the condition,

$$\kappa(e_{-n}^\perp v) = [e_{-n}^\perp, \kappa(v)]_{q^{-n}} - q^{-n} \sum_{j \geq 1} \kappa(\psi_j^{+, \perp} v) e_{-n-j}^\perp,$$

for all  $v \in V$ ,  $n > 0$ , cf. (4.2). Note that the sum on the right hand side is finite for any  $v$ .

Then  $N$  is well defined and Corollary 3.5 still holds.

The vacuum to vacuum matrix element of  $L$  operator now is just the product of operators in (4.8),  $L_{\emptyset, \emptyset}(\mathbf{u}) = \prod_{i=1}^k L_{\emptyset, \emptyset}(u_i)$ . The vacuum eigenvalue  $\gamma_r$ , cf. (2.20), for  $V$  reads  $\gamma_r = \frac{1-q_2^r}{r\kappa_r} \sum_{i=1}^k u_i^r$ . With this change, Lemma 4.1 remains valid. We use the operator  $L_{\emptyset, \emptyset}(\mathbf{u})$  to define the map  $\iota$  as in the case of  $k = 1$ .

We define the space of  $p$  commutators  $J_p$  in the same way. And then Theorem 5.2 holds just the same. We introduce the Bethe equation

$$(6.1) \quad 1 = q^{-k} p \cdot \prod_{j=1}^k \frac{a_i - q_2 u_j}{a_i - u_j} \prod_{j(\neq i)} \frac{(a_j - q_1 a_i)(a_j - q_2 a_i)(a_j - q_3 a_i)}{(a_j - q_1^{-1} a_i)(a_j - q_2^{-1} a_i)(a_j - q_3^{-1} a_i)}, \quad i = 1, \dots, n.$$

and arrive at the generalization of Theorem 5.4.

**Theorem 6.2.** *Let  $a = (a_1, \dots, a_n)$  be a solution of (6.1) such that  $ev_a$  is non-zero. Then the restriction of  $ev_a$  to  $N = \kappa(V)$  is an eigenvector in  $V^*$  of the first integral of motion  $\tilde{e}_0^\perp(p)$  (5.2) with the eigenvalue*

$$E_1(a) = - \sum_{i=1}^n a_i + \frac{\sum_{i=1}^k u_i}{(1 - q_1)(1 - q_3)}.$$

For generic  $p$ ,  $ev_a$  is a joint eigenvector of  $\{I_{\mathcal{F}, n}(p)\}_{n=1}^\infty$ .

## APPENDIX A. GORDON FILTRATION

In this section we study the size of the space  $Sh_1(u)$  using the technique of the Gordon filtration. Our goal is to prove Corollary A.2 below.

Let  $n$  be a positive integer, and let  $\lambda$  be a partition such that  $|\lambda| \leq n$ . For an element  $F \in Sh_{1, n}(u)$ , we introduce an operation of specialization  $\rho_\lambda(F)$  as follows.

First we set

$$\rho_\lambda^{(0)}(F)(y_1, \dots, y_{\ell(\lambda)}, x_{|\lambda|+1}, \dots, x_n) = F(x_1, \dots, x_n) \Big|_{x_{\lambda_1 + \dots + \lambda_{i-1} + j} = q_1^{j-1} y_i \quad (1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i)}.$$

The wheel condition (3.1) implies that  $\rho_\lambda^{(0)}(F)$  is divisible by the factor

(A.1)

$$\prod_{1 \leq a < b \leq \ell(\lambda)} \prod_{\substack{1 \leq i \leq \lambda_a - 1 \\ 1 \leq j \leq \lambda_b}} (q_1^{j-1} y_b - q_1^i q_2 y_a) (q_1^{j-1} y_b - q_1^i q_3 y_a) \prod_{k=|\lambda|+1}^n \prod_{\substack{1 \leq i \leq \ell(\lambda) \\ 1 \leq j \leq \lambda_i - 1}} (x_k - q_1^j q_2 y_i) (x_k - q_1^j q_3 y_i).$$

Next we set

$$\rho_\lambda^{(1)}(F)(y_2, \dots, y_{\ell(\lambda)}, x_{|\lambda|+1}, \dots, x_n) = \left[ (y_1 - u) \rho_\lambda^{(0)}(F)(y_1, \dots, y_{\ell(\lambda)}, x_{|\lambda|+1}, \dots, x_n) \right] \Big|_{y_1 = u}.$$

Then  $\rho_\lambda^{(1)}(F)$  is divisible further by the factor

$$\prod_{k=|\lambda|+1}^n (x_k - q_2 u)$$

due to the wheel conditions (3.1) and (3.2). For  $i \geq 2$ , we remove a factor contained in (A.1) to define

$$\begin{aligned} & \rho_\lambda^{(i)}(F)(y_{i+1}, \dots, y_{\ell(\lambda)}, x_{|\lambda|+1}, \dots, x_n) \\ &= \left[ (y_i - q_3^{i-1}u)^{-\lambda_i+1} \rho_\lambda^{(i-1)}(F)(y_i, \dots, y_{\ell(\lambda)}, x_{|\lambda|+1}, \dots, x_n) \right] \Big|_{y_i=q_3^{i-1}u}. \end{aligned}$$

Finally we set  $\rho_\lambda(F) = \rho_\lambda^{(\ell(\lambda))}(F)$ .

At each step, the wheel condition produces further factors. Collecting them together, we find that

$$(A.2) \quad \rho_\lambda(F) \in Sh_{0,n-|\lambda|} \times \prod_{k=|\lambda|+1}^n \left[ \prod_{(i,j) \in \lambda} \omega(x_k, q_3^{i-1}q_1^{j-1}u) \times \frac{1}{\prod_{(i,j) \in CC(\lambda)} (x_k - q_3^{i-1}q_1^{j-1}u)} \right].$$

Let  $\mathcal{P}_{\leq n}$  denote the set of all partitions  $\lambda$  with  $|\lambda| \leq n$ . Define a total ordering  $>$  on  $\mathcal{P}_{\leq n}$  by setting  $\mu > \lambda$  iff there is a  $k$  such that  $\mu_1 = \lambda_1, \dots, \mu_{k-1} = \lambda_{k-1}$ ,  $\mu_k > \lambda_k$ . We introduce a decreasing filtration  $\{V_{n,\lambda}\}_{\lambda \in \mathcal{P}_{\leq n}}$  on the space  $V_n = Sh_{1,n}(u)$  by setting

$$V_{n,\lambda} = \bigcap_{\mu > \lambda} \text{Ker } \rho_\mu \subset V_n.$$

**Proposition A.1.** *If  $|\lambda| > m$  then  $\rho_\lambda(Sh_{1,m}(u) * Sh_{0,n-m}) = 0$ .*

*If  $|\lambda| = m$  then  $\rho_\lambda(Sh_{1,m}(u) * Sh_{0,n-m}) = \rho_\lambda(V_{n,\lambda})$ .*

*If  $|\lambda| = n$  then  $\rho_\lambda(V_{n,\lambda}) = \mathbb{C}$ .*

*Proof.* The first statement is straightforward as all terms in the symmetrization of the product  $Sh_{1,m}(u) * Sh_{0,n-m}$  clearly vanish under evaluation  $\rho_\lambda$  if  $|\lambda| > m$ .

Let  $m = |\lambda|$ . From the definition of the space  $V_{n,\lambda}$  we see that, if  $F$  is an element of  $V_{n,\lambda}$ , then  $\rho_\lambda(F)$  contains an extra factor  $\prod_{k=m+1}^n \prod_{(i,j) \in CC(\lambda)} (x_k - q_3^{i-1}q_1^{j-1}u)$  which cancels the denominator of (A.2). Therefore we have

$$(A.3) \quad \rho_\lambda(V_{n,\lambda}) \subset Sh_{0,n-m} \times \prod_{k=m+1}^n \left[ \prod_{(i,j) \in \lambda} \omega(x_k, q_3^{i-1}q_1^{j-1}u) \right].$$

On the other hand, the following identity holds for elements  $G \in Sh_{1,m}(u)$  and  $H \in Sh_{0,n-m}$ :

$$(A.4) \quad \rho_\lambda(G * H) = \text{const. } \rho_\lambda(G) \cdot H(x_{m+1}, \dots, x_n) \times \prod_{\substack{m+1 \leq k \leq n \\ (i,j) \in \bar{\lambda}}} \omega(x_k, q_3^{i-1}q_1^{j-1}u),$$

where *const.* is non-zero. We choose

$$G = \epsilon_{\lambda'}^{(q_1)}(x) \times \prod_{i=1}^m \frac{x_i - q_2 u}{x_i - u} \in Sh_{1,m}(u),$$

where  $\lambda' = (\lambda'_1, \dots, \lambda'_{\ell'})$  is the partition dual to  $\lambda$  and  $\epsilon_{\lambda'}^{(q_1)}(x)$  is defined in (5.6) where the parameter  $q_3$  is changed to  $q_1$ . It is easy to see that  $G \in V_{m,\lambda}$ , and that  $\rho_\lambda(G)$  is a non-vanishing complex number. Since  $H \in Sh_{0,n-m}$  is arbitrary in (A.4), we conclude that the inclusion in (A.3) is actually an equality and that  $\rho_\lambda(V_{n,\lambda}) = \rho_\lambda(Sh_{1,m}(u) * Sh_{0,n-m})$ .

In particular, if  $m = n$ , then  $\rho_\lambda(V_{n,\lambda})$  is a one dimensional vector space.  $\square$

**Corollary A.2.** *The space  $Sh_{1,n}(u)/(\sum_{m=0}^{n-1} Sh_{1,m}(u) * Sh_{0,n-m})$  is a finite dimensional vector space of dimension  $p(n)$ , the number of partitions of  $n$ .*

*Proof.* We consider the associated graded space related to the filtration  $\{V_{n,\lambda}\}$  of the space  $V_n = Sh_{1,n}(u)$ . By the definition, we have  $\text{gr}_\lambda(V_n) = \rho_\lambda(V_{n,\lambda})$ . Now, for the factor space, from Proposition A.1 we have  $\text{gr}_\lambda(Sh_{1,n}(u)/(\sum_{m=0}^{n-1} Sh_{1,m}(u) * Sh_{0,n-m}))$  is zero if  $|\lambda| < n$  and one dimensional if  $|\lambda| = n$ . The corollary follows.  $\square$

**Acknowledgments.** We would like to thank Jun'ichi Shiraishi for enlightening discussions.

The financial support from the Government of the Russian Federation within the framework of the implementation of the 5-100 Programme Roadmap of the National Research University Higher School of Economics is acknowledged. Research of MJ is supported by the Grant-in-Aid for Scientific Research B-23340039.

EM and BF would like to thank Kyoto University and Rikkyo University for hospitality during their visits when the main part of this work was completed.

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